

Rate and Cross section

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} \text{ J} \cdot \text{s}, c = 2.998 \times 10^8 \text{ m/s}, 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}, \hbar c = 197 \text{ MeV fm}, k_B = 8.617 \times 10^{-5} \text{ eV/K}$$

$$1 \text{ GeV} = 10^3 \text{ MeV} = 10^6 \text{ keV} = 10^9 \text{ eV}, 1 \text{ m} = 100 \text{ cm} = 10^9 \text{ nm} = 10^{15} \text{ fm}$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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1 Cross section

Here, we will explore the strength and the properties of interactions by comparing the probabilistic amplitudes with the results from the actual experiments. In the experimental setup, we either collide two particle beams or scatter a beam by a fixed target. In such a case, we expect multiple collisions. Therefore, we cannot derive the correct information without considering the precise setup, e.g., beam strength.

Motivation of introducing cross section

In practice, we gain various information about elementary particles and interactions from the flux of the scattered/decay particles. The schematic image of the flux is given in Fig. 1.

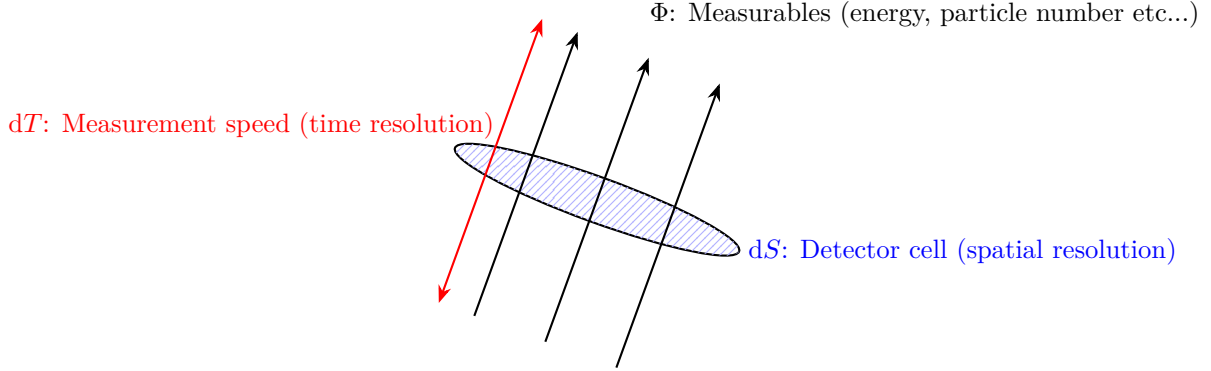


Figure 1: Schematic picture of practical setup in particle physics experiment

The unit of flux in SI/natural unit is summarized in the following table.

	Energy flux	Number flux	Luminosity (Power)
In SI unit	$\text{J} / (\text{m}^2 \cdot \text{s})$	$\# / (\text{m}^2 \cdot \text{s})$	$\text{J/s} = \text{W}$
In natural unit	GeV^4	GeV^3	GeV^2

As an example, let us consider a fixed target experiment. The setup is given in Fig. 2. The event rate is proportional to the number flux of A:

$$\begin{aligned} \text{Rate} &= \frac{\#}{(\text{time})} \\ &\propto (\text{number flux of A}) \\ &= n_A v_A \end{aligned} \tag{1}$$

$$\Rightarrow \frac{\#}{(\text{time})} = n_A v_A \sigma_B. \tag{2}$$

Here, σ_B has a unit of “area.” Therefore, σ_B is called “total cross section.”

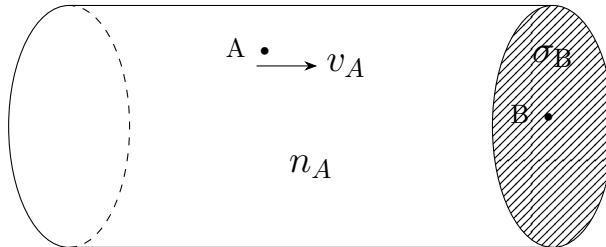


Figure 2: The setup of the fixed target experiment with particles A & B

Basic assumptions of beams

First, we list three basic assumptions of particle beams

1. The characteristic size of the beam is large enough. Even though it may look collimated compared to our ordinary sense, you can approximate its size to be infinitely large compared to the characteristic size of the beam's particles.
2. Inside the initial beams, each beam particle is separate enough that you can neglect the internal interactions between them. Therefore, any particles can be described as free (no interactions) when beams are separate.
3. Whole processes take an infinitely long time compared to the typical timescale of each microscopic process, even if the whole processes may look instantaneous because incoming initial particles and outgoing particles are ultra-fast.

Definition of cross section

A *cross section* of a collision process is a quantity which is given by

$$\frac{(\text{Number of collisions per unit volume per unit time})}{(\text{Particle number density of beam 1})(\text{Particle number density of beam 2})(\text{Relative speed of two beams})}. \quad (3)$$

Suppose the collisions of two incident beams with number density ρ_1 and ρ_2 , respectively. Let v_{rel} be relative speed. When we measure that the N collisions happen within the space volume of size V and time interval T , the cross section σ of these collisions is given by

$$\sigma = \frac{1}{\rho_1 \rho_2 v_{\text{rel}}} \left(\frac{N}{VT} \right) \quad (4)$$

When we are focusing on a special final state, such as a state which is scattered within the angle $(\theta, \theta + d\theta)$, we write

$$d\sigma = \frac{1}{\rho_1 \rho_2 v_{\text{rel}}} \left(\frac{dN}{VT} \right), \quad (5)$$

where $d\sigma$ or $d\sigma/d\theta$ is called a *differential cross section* in the direction of θ .

S matrix formalism

S matrix or scattering matrix is a matrix that relates the initial state and the final state of a physical system undergoing a scattering process. It is introduced by the transition amplitude of two asymptotic states before ($t \rightarrow -\infty$) and after ($t \rightarrow \infty$) the scattering process: in-state $\rightarrow$ out-state

$$S_{\beta\alpha} = \langle \Psi_\beta(\infty) | \Phi_\alpha(-\infty) \rangle, \quad (6)$$

where Ψ_β belongs to the free final state Hilbert space $\mathcal{H}_{out}$, and Φ_α belongs to the free initial state Hilbert space $\mathcal{H}_{in}$. Usually, $\mathcal{H}_{out} \neq \mathcal{H}_{in}$. Here α and β are shorthands that represent the particle content but suppresses the individual labels. We can proceed further to get a *S operator*

$$S_{\beta\alpha} = \langle \Psi_\beta(\infty) | \Phi_\alpha(-\infty) \rangle = \langle \Psi_\beta(-\infty) | \hat{S} | \Phi_\alpha(-\infty) \rangle = \langle \Psi_\beta(+\infty) | \hat{S} | \Phi_\alpha(+\infty) \rangle. \quad (7)$$

Now the final state is also expressed in the form of S operator acting on some state in $\mathcal{H}_{in}$.

The concept of asymptotic states ($t \rightarrow \pm\infty$) works very well under our assumption (2) and (3) of the beam.

Let us express cross sections via the S matrix. Suppose particles 1 and 2 with momentum

$$p_{1,2}^\mu = \begin{pmatrix} p_{1,2}^0 \\ \mathbf{p}_{1,2} \end{pmatrix} \quad (8)$$

collide with each other and the final state is composed of particles of mass M_i with momentum $\mathbf{q}_i \sim \mathbf{q}_i + d^3\mathbf{q}_i$.

(1) Relative speed

The relative speed of the incident beams is given by

$$\begin{aligned} v_{\text{rel}} &\equiv |\mathbf{v}_1 - \mathbf{v}_2| \\ &= \left| \frac{\mathbf{p}_1}{p_1^0} - \frac{\mathbf{p}_2}{p_2^0} \right| \\ &= \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1^0 p_2^0}. \end{aligned} \quad (9)$$

(2) Number density

Using 1 particle wavefunction, $\langle \mathbf{p} | \mathbf{p} \rangle$ is evaluated by

$$\begin{aligned} \langle \mathbf{p} | \mathbf{p} \rangle &= \int d^3\mathbf{x} \psi_{\mathbf{p}}^*(\mathbf{x}) \psi_{\mathbf{p}}(\mathbf{x}) \\ &= \int d^3\mathbf{x} |\psi_{\mathbf{p}}(\mathbf{x})|^2, \end{aligned} \quad (10)$$

and this means that $\langle \mathbf{p} | \mathbf{p} \rangle$ reflects the total number of particles with momentum $\mathbf{p}$ in the whole spatial volume V . Using the normalization of $|\mathbf{p}\rangle$

$$\langle \mathbf{p} | \mathbf{p}' \rangle = (2\pi)^3 2p^0 \delta^3(\mathbf{p} - \mathbf{p}'), \quad (11)$$

we get

$$\begin{aligned} \langle \mathbf{p} | \mathbf{p} \rangle &= (2\pi)^3 2p^0 \delta^3(\mathbf{0}) \\ &= 2p^0 \int d^3\mathbf{x} e^{i\mathbf{p} \cdot \mathbf{x}} \Big|_{\mathbf{p}=\mathbf{0}} \\ &= 2p^0 \int d^3\mathbf{x} \\ &= 2p^0 V. \end{aligned} \quad (12)$$

Thus, the number density of the particle with momentum $\mathbf{p}$ is given by

$$\rho(\mathbf{p}) = 2p^0. \quad (13)$$

We can use this expression for the number density of the initial state.

(3) Scattering probability

For the final state, we need to think of the general multiparticle states. n' particles state

$$|\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}\rangle \quad (14)$$

now has the norm of

$$\langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} \rangle = (2V)^{n'} k_1^0 k_2^0 \dots k_{n'}^0. \quad (15)$$

When we consider the state subspace such that the set of the final momentum $\{\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}\}$ belongs to the infinitesimal momentum space element $\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0}$, the projection operator to that state subspace is given by

$$\mathcal{P} \equiv \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) |\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}\rangle \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}|. \quad (16)$$

From Eq. (7), the initial state $|\alpha\rangle$ is transformed into $\hat{S}|\alpha\rangle$ after the interactions. Thus, the projection of this state onto the infinitesimal momentum space element above reads

$$\mathcal{P}(\hat{S}|\alpha\rangle) = \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) |\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}\rangle \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} |\alpha\rangle. \quad (17)$$

Following the postulate of the quantum mechanics (Born's rule), the probability of finding the final state $|\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}\rangle$ in the infinitesimal momentum space element $\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0}$ by the single collision is given by the squared norm of the Eq. (17) divided by the squared norm of the initial state:

$$\begin{aligned} dP &= \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}'_i}{(2\pi)^3 2k_i^0} \right) \frac{\left(\langle \mathbf{k}'_1, \mathbf{k}'_2, \dots, \mathbf{k}'_{n'} | \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} \rangle \langle \mathbf{k}'_1, \mathbf{k}'_2, \dots, \mathbf{k}'_{n'} | \hat{S} |\alpha\rangle \right) \left(\langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} |\alpha\rangle \right)}{\langle \alpha | \alpha \rangle \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} \rangle} \\ &= \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \frac{\left| \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} |\alpha\rangle \right|^2}{\langle \alpha | \alpha \rangle}. \end{aligned} \quad (18)$$

(4) $2 \rightarrow n'$ scattering

Consider the case when the 2 initial particles ($|\alpha\rangle = |\mathbf{p}_1, \mathbf{p}_2\rangle$) interact to create n' particles. Then

$$\langle \alpha | \alpha \rangle = 4V^2 p_1^0 p_2^0. \quad (19)$$

In the spatial volume V , the number of the initial particles 1 and 2 are $\rho_1 V, \rho_2 V$, respectively. Then, the total number of particles in the final state $|\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}\rangle$ is given by

$$\begin{aligned} dN &= (\rho_1 V) (\rho_2 V) dP \\ &= \frac{V^2 \rho_1 \rho_2}{\langle \alpha | \alpha \rangle} \left| \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} |\alpha\rangle \right|^2 \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \\ &= \frac{\rho_1 \rho_2}{4p_1^0 p_2^0} \left| \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} |\alpha\rangle \right|^2 \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \end{aligned} \quad (20)$$

(5) S matrix element

Total energy-momentum must be conserved before and after the interactions. Then $P_{\text{before}} = p_1 + p_2$ and $P_{\text{after}} = k_1 + k_2 + \dots + k_{n'}$ are equivalent. Thus the S matrix element must be in the form of

$$\langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} | \alpha \rangle = i (2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \mathfrak{M}, \quad (21)$$

where Lorentz invariant $\mathfrak{M}$ is the invariant scattering amplitude. Then

$$\begin{aligned} \left| \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} | \alpha \rangle \right|^2 &= |\mathfrak{M}|^2 \left[(2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \right]^2 \\ &= |\mathfrak{M}|^2 \left[(2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \right] \left[(2\pi)^4 \delta^4(0) \right] \\ &= |\mathfrak{M}|^2 \left[(2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \right] \left(\int d^4x e^{-ik \cdot x} \Big|_{k=0} \right) \\ &= VT |\mathfrak{M}|^2 (2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \end{aligned} \quad (22)$$

Cross section

From the previous discussion, we get

$$\begin{aligned} dN &= \frac{\rho_1 \rho_2}{4p_1^0 p_2^0} \left| \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} | \alpha \rangle \right|^2 \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \\ &= \frac{\rho_1 \rho_2}{4p_1^0 p_2^0} VT |\mathfrak{M}|^2 (2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \\ &\equiv \frac{\rho_1 \rho_2}{4p_1^0 p_2^0} VT |\mathfrak{M}|^2 d\text{LIPS}_{n'}, \end{aligned} \quad (23)$$

where $d\text{LIPS}_{n'}$ is the n' -body Lorentz-invariant phase-space measure

$$d\text{LIPS}_{n'} \equiv (2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \quad (24)$$

Using the definition of the cross section (5), you get

$$\begin{aligned} d\sigma &= \frac{1}{\rho_1 \rho_2 v_{\text{rel}}} \left(\frac{dN}{VT} \right) \\ &= \frac{1}{4p_1^0 p_2^0} \left(\frac{p_1^0 p_2^0}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \right) |\mathfrak{M}|^2 d\text{LIPS}_{n'} \left(v_{\text{rel}} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1^0 p_2^0} \right) \\ &= \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |\mathfrak{M}|^2 d\text{LIPS}_{n'} \end{aligned} \quad (25)$$

2 Decay rate

(1) $1 \rightarrow n'$ decay

Consider the case when the 1 initial particle ($|\alpha\rangle = |\mathbf{p}\rangle$) decays into n' particles. Then

$$\langle\alpha|\alpha\rangle = 2Vp^0. \quad (26)$$

In the spatial volume V , the number of the initial particles is ρV , respectively. Then, the total number of particles in the final state $|\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'}\rangle$ is given by

$$\begin{aligned} dN &= (\rho V) dP \\ &= \frac{V\rho}{\langle\alpha|\alpha\rangle} \left| \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} | \alpha \rangle \right|^2 \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \\ &= \frac{\rho}{2p^0} \left| \langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n'} | \hat{S} | \alpha \rangle \right|^2 \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \\ &= \frac{\rho}{2p^0} VT |\mathfrak{M}|^2 (2\pi)^4 \delta^4(P_{\text{before}} - P_{\text{after}}) \left(\prod_{i=1}^{n'} \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2k_i^0} \right) \\ &= \frac{\rho}{2p^0} VT |\mathfrak{M}|^2 d\text{LIPS}_{n'} \end{aligned} \quad (27)$$

(2) Decay width

The differential decay width is given by

$$\begin{aligned} d\Gamma &= \frac{dN}{\rho VT} \\ &= \frac{1}{2p^0} |\mathfrak{M}|^2 d\text{LIPS}_{n'}. \end{aligned} \quad (28)$$

When we consider the rest frame of the initial particles, the differential decay width is given by

$$d\Gamma = \frac{1}{2M} |\mathfrak{M}|^2 d\text{LIPS}_{n'}, \quad (29)$$

where M is the rest mass of the initial particle.

3 Compton Scattering

Initial and Final States

The initial states are given by:

$$e^- : p^\mu = \begin{pmatrix} E \\ \mathbf{p} \end{pmatrix}, \quad \text{with spinor } u = u_r(\mathbf{p}),$$

$$\gamma : k^\mu = \begin{pmatrix} \omega \\ \mathbf{k} \end{pmatrix}, \quad \text{with polarization vector } \varepsilon^\mu = \varepsilon_s^\mu(\mathbf{k})$$

The final states are given by:

$$e^- : p'^\mu = \begin{pmatrix} E' \\ \mathbf{p}' \end{pmatrix}, \quad \text{with spinor } u' = u_{r'}(\mathbf{p}'),$$

$$\gamma : k'^\mu = \begin{pmatrix} \omega' \\ \mathbf{k}' \end{pmatrix}, \quad \text{with polarization vector } \varepsilon'^\mu = \varepsilon_{s'}^\mu(\mathbf{k}')$$

The momentum conservation reads:

$$p^\mu + k^\mu = p'^\mu + k'^\mu. \quad (30)$$

By squaring both sides, we get

$$(p \cdot k') = (p' \cdot k), \quad (p \cdot k) = (p' \cdot k'). \quad (31)$$

where we used the on-shell condition

$$p^2 = p'^2 = m^2, \quad k^2 = k'^2 = 0. \quad (32)$$

Tree-Level Scattering Amplitude

The tree diagrams contributing to this process are given by Fig. 3. Defining

$$f_1 \equiv p + k, \quad f_2 \equiv p - k', \quad (33)$$

we obtain the scattering amplitude $\mathfrak{M}$ by the sum of two terms:

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2, \quad (34)$$

where

$$i\mathfrak{M}_1 = (-e^2) \frac{\bar{u}' \not{\varepsilon}' (f_1 + m) \not{\varepsilon} u}{2(p \cdot k)}, \quad (35)$$

$$i\mathfrak{M}_2 = (-e^2) \frac{\bar{u}' \not{\varepsilon} (f_2 + m) \not{\varepsilon}' u}{2(p \cdot k')}. \quad (36)$$

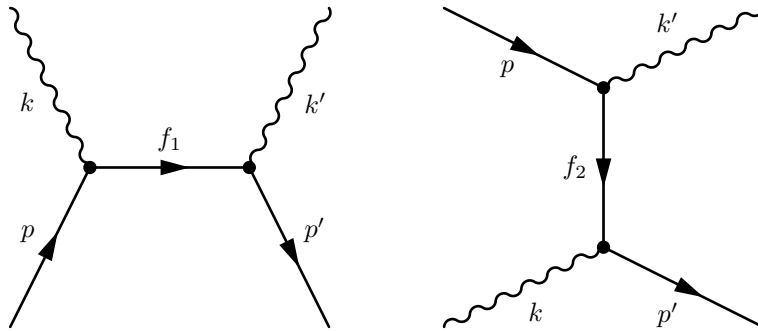


Figure 3: The tree diagrams of the Compton scattering

Computation of the Amplitude Squared

The amplitude squared is given by

$$\begin{aligned} |\mathfrak{M}|^2 &= |\mathfrak{M}_1 + \mathfrak{M}_2|^2 \\ &= |\mathfrak{M}_1|^2 + |\mathfrak{M}_2|^2 + \mathfrak{M}_1 \mathfrak{M}_2^* + \mathfrak{M}_2 \mathfrak{M}_1^*. \end{aligned} \quad (37)$$

Let $\mathfrak{M}_{\alpha\beta}$ be polarization independent amplitude, which satisfies

$$\mathfrak{M} = \varepsilon_s^\alpha \varepsilon_{s'}'^\beta \mathfrak{M}_{\alpha\beta}. \quad (38)$$

Then we can perform the spin/polarization average of this amplitude for initial and final states as

$$\begin{aligned} \overline{|\mathfrak{M}|^2} &= \frac{1}{4} \sum_{s,s'} \sum_{r,r'} |\mathfrak{M}|^2 \\ &= \frac{1}{4} \sum_{s,s'} \sum_{r,r'} \varepsilon_s^\alpha \varepsilon_{s'}'^\beta \varepsilon_s^\gamma \varepsilon_{s'}'^\delta \mathfrak{M}_{\alpha\beta} \mathfrak{M}_{\gamma\delta}^* \\ &= \frac{1}{4} \sum_{r,r'} g^{\alpha\gamma} g^{\beta\delta} \mathfrak{M}_{\alpha\beta} \mathfrak{M}_{\gamma\delta}^* \quad (\because k^\alpha k^\beta \mathfrak{M}_{\alpha\beta} = 0) \\ &= \frac{1}{4} \sum_{r,r'} \mathfrak{M}^{\alpha\beta} \mathfrak{M}_{\alpha\beta}^* \end{aligned} \quad (39)$$

Substituting the full expression of $\mathfrak{M}$ and taking the spin sum, you get the four sets of traces as

$$\begin{aligned} \overline{|\mathfrak{M}|^2} &= \frac{1}{4} \sum_{r,r'} \mathfrak{M}^{\alpha\beta} \mathfrak{M}_{\alpha\beta}^* \\ &= \frac{1}{4} \left(\frac{e^4}{4} \right) \left\{ \frac{1}{(p \cdot k)^2} \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\alpha (\not{f}_1 + m) \gamma_\beta (\not{p}' + m)] \right. \\ &\quad + \frac{1}{(p \cdot k')^2} \text{Tr} [\gamma^\alpha (\not{f}_2 + m) \gamma^\beta (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \\ &\quad + \frac{1}{(p \cdot k)(p \cdot k')} \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \\ &\quad \left. + \frac{1}{(p \cdot k)(p \cdot k')} \text{Tr} [\gamma^\alpha (\not{f}_1 + m) \gamma^\beta (\not{p} + m) \gamma_\alpha (\not{f}_2 + m) \gamma_\beta (\not{p}' + m)] \right\} \\ &= \frac{1}{4} \left(\frac{e^4}{4} \right) \left\{ \frac{1}{(p \cdot k)^2} \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\alpha (\not{f}_1 + m) \gamma_\beta (\not{p}' + m)] \right. \\ &\quad + \frac{1}{(p \cdot k')^2} \text{Tr} [\gamma^\alpha (\not{f}_2 + m) \gamma^\beta (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \\ &\quad \left. + \frac{2}{(p \cdot k)(p \cdot k')} \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \right\} \end{aligned} \quad (40)$$

Since

$$\begin{aligned} \gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\alpha (\not{f}_1 + m) \gamma_\beta &= \gamma^\beta (\not{f}_1 + m) (-\not{p} \gamma^\alpha + 2p^\alpha + m \gamma^\alpha) \gamma_\alpha (\not{f}_1 + m) \gamma_\beta \\ &= \gamma^\beta (\not{f}_1 + m) (-2\not{p} + 4m) (\not{f}_1 + m) \gamma_\beta \\ &= 4\not{f}_1 \not{p} \not{f}_1 + m [-16(p \cdot f_1) + 16f_1^2] + m^2 (4\not{p} - 16\not{f}_1) + 16m^3, \end{aligned} \quad (41)$$

the first trace is evaluated by

$$\begin{aligned}
& \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\alpha (\not{f}_1 + m) \gamma_\beta (\not{p}' + m)] \\
&= \text{Tr} [\{4\not{f}_1 \not{p} \not{f}_1 + m [-16(p \cdot f_1) + 16f_1^2] + m^2 (4\not{p} - 16\not{f}_1) + 16m^3\} (\not{p}' + m)] \\
&= 4f_{1\alpha} p_\beta f_{1\gamma} p'_\delta \text{Tr} [\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta] + m^2 (4p_\alpha p'_\beta - 16f_{1\alpha} p'_\beta) \text{Tr} [\gamma^\alpha \gamma^\beta] + 4m^2 [-16(p \cdot f_1) + 16f_1^2] + 64m^4 \\
&= [32(p \cdot f_1)(p' \cdot f_1) - 16f_1^2(p \cdot p')] + m^2 [16(p \cdot p') - 64(p' \cdot f_1)] + 4m^2 [-16(p \cdot f_1) + 16f_1^2] + 64m^4 \\
&= 16m^2(p \cdot p') + 32m^2(p' \cdot k) + 32(p \cdot k)(p' \cdot k) - 48m^2(p \cdot p') - 64m^2(p' \cdot k) + 64m^2(p \cdot k) + 64m^4 \\
&= 32m^4 + 32m^2(p \cdot k) + 32(p \cdot k)(p' \cdot k) \\
&= 32[m^4 + m^2(p \cdot k) + (p \cdot k)(p' \cdot k)].
\end{aligned} \tag{42}$$

We evaluate the second trace in the same way as

$$\begin{aligned}
& \text{Tr} [\gamma^\alpha (\not{f}_2 + m) \gamma^\beta (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \\
&= \text{Tr} [\{4\not{f}_2 \not{p} \not{f}_2 + m [-16(p \cdot f_2) + 16f_2^2] + m^2 (4\not{p} - 16\not{f}_2) + 16m^3\} (\not{p}' + m)] \\
&= 4f_{2\alpha} p_\beta f_{2\gamma} p'_\delta \text{Tr} [\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta] + m^2 (4p_\alpha p'_\beta - 16f_{2\alpha} p'_\beta) \text{Tr} [\gamma^\alpha \gamma^\beta] + 4m^2 [-16(p \cdot f_2) + 16f_2^2] + 64m^4 \\
&= [32(p \cdot f_2)(p' \cdot f_2) - 16f_2^2(p \cdot p')] + m^2 [16(p \cdot p') - 64(p' \cdot f_2)] + 4m^2 [-16(p \cdot f_2) + 16f_2^2] + 64m^4 \\
&= 32[m^4 - m^2(p \cdot k') + (p \cdot k)(p \cdot k')].
\end{aligned} \tag{43}$$

For the third trace, we obtain

$$\begin{aligned}
& \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \\
&= \text{Tr} [\{-2\not{p} \gamma^\alpha \not{f}_1 + 4m(p^\alpha + k^\alpha) - 2m^2 \gamma^\alpha\} (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \\
&= -2p_\mu f_{1\nu} f_{2\rho} p'_\sigma \text{Tr} [\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\rho \gamma_\alpha \gamma^\sigma] - 2m^2 (p_\mu f_{1\nu} + f_{2\nu} p'_\mu) \text{Tr} [\gamma^\mu \gamma^\alpha \gamma^\nu \gamma_\alpha] \\
&\quad - 2m^4 \text{Tr} [\gamma^\alpha \gamma_\alpha] + 4m(2p^\alpha + k^\alpha) p'^\mu \text{Tr} [\gamma_\alpha \gamma_\mu] \\
&= -16m^4 - 16m^2(p \cdot k) + 32m^2(p \cdot k') + 64m^4 \\
&= -16m^4 - 16m^2(p \cdot k) + 32m^2(p \cdot k') + 64m^4 + 32m^2(p \cdot k) - 32m^2(p \cdot k') - 16m^4 - 16m^2(p \cdot k') \\
&= 32m^4 + 16m^2(p \cdot k) - 16m^2(p \cdot k').
\end{aligned} \tag{44}$$

Then, from Eq. (42), (43), and (44), we get

$$\begin{aligned}
|\overline{\mathfrak{M}}|^2 &= \frac{1}{4} \left(\frac{e^4}{4} \right) \left\{ \frac{1}{(p \cdot k)^2} \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\alpha (\not{f}_1 + m) \gamma_\beta (\not{p}' + m)] \right. \\
&\quad + \frac{1}{(p \cdot k')^2} \text{Tr} [\gamma^\alpha (\not{f}_2 + m) \gamma^\beta (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \\
&\quad \left. + \frac{2}{(p \cdot k)(p \cdot k')} \text{Tr} [\gamma^\beta (\not{f}_1 + m) \gamma^\alpha (\not{p} + m) \gamma_\beta (\not{f}_2 + m) \gamma_\alpha (\not{p}' + m)] \right\} \\
&= \frac{e^4}{16} \left\{ 32 \left[\frac{m^4}{(p \cdot k)^2} + \frac{m^2}{(p \cdot k)} + \frac{(p' \cdot k)}{(p \cdot k)} \right] + 32 \left[\frac{m^4}{(p \cdot k')^2} + \frac{m^2}{(p \cdot k')} + \frac{(p \cdot k)}{(p \cdot k')} \right] + \frac{64m^4}{(p \cdot k)(p \cdot k')} + \frac{32m^2}{(p \cdot k')} - \frac{32m^2}{(p \cdot k)} \right\} \\
&= 2e^4 \left\{ \left[\frac{(p' \cdot k)}{(p \cdot k)} + \frac{(p \cdot k)}{(p \cdot k')} \right] + 2m^2 \left[\frac{1}{(p \cdot k)} - \frac{1}{(p \cdot k')} \right] + m^4 \left[\frac{1}{(p \cdot k)} - \frac{1}{(p \cdot k')} \right]^2 \right\}
\end{aligned} \tag{45}$$

This form is independent of the choice of inertial frame. From now we fix our frame to the rest frame of the initial electron. This choice fully simplifies the form.

Klein-Nishina Formula

In the rest frame of the initial electron,

$$p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow (p \cdot k) = m\omega, \quad (p \cdot k') = (p' \cdot k) = m\omega'. \quad (46)$$

Let (θ, φ) be the polar angles of $\mathbf{k}'$ and

$$d\Omega = \sin \theta \, d\theta d\varphi \quad (47)$$

is the corresponding element of solid angle. We take $\mathbf{k}$ as polar coordinate axis so that θ is the photon scattering angle *i.e.*

$$\mathbf{k} \cdot \mathbf{k}' = \omega\omega' \cos \theta. \quad (48)$$

Momentum conservation implies

$$\begin{aligned} (p \cdot k) &= (p' \cdot k) + (k' \cdot k) \\ \Rightarrow m\omega &= m\omega' + \omega\omega' - \omega\omega' \cos \theta \\ \therefore \omega' &= \frac{m\omega}{m + \omega(1 - \cos \theta)} \end{aligned} \quad (49)$$

Therefore, in this frame, $\overline{|\mathfrak{M}|^2}$ is simplified to

$$\begin{aligned} \overline{|\mathfrak{M}|^2} &= 2e^4 \left\{ \left[\frac{(p' \cdot k)}{(p \cdot k)} + \frac{(p \cdot k)}{(p \cdot k')} \right] + 2m^2 \left[\frac{1}{(p \cdot k)} - \frac{1}{(p \cdot k')} \right] + m^4 \left[\frac{1}{(p \cdot k)} - \frac{1}{(p \cdot k')} \right] \right\} \\ &= 2e^4 \left[\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} \right) + 2m \left(\frac{1}{\omega} + \frac{1}{\omega'} \right) + m^2 \left(\frac{1}{\omega} - \frac{1}{\omega'} \right)^2 \right] \\ &= 2e^4 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \end{aligned} \quad (50)$$

The relative speed of the incident beams is given by

$$v_{\text{rel}} = \frac{\sqrt{(p \cdot k)^2 - m^2 \, 0^2}}{p^0 k^0} = 1. \quad (51)$$

The differential cross section is given by

$$\begin{aligned} d\sigma &= \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \overline{|\mathfrak{M}|^2} d\text{LIPS}_{2'} \\ &= \frac{1}{4m\omega} \overline{|\mathfrak{M}|^2} d\text{LIPS}_{2'} \end{aligned} \quad (52)$$

where $d\text{LIPS}_{n'}$ is the n' -body Lorentz-invariant phase-space measure

$$d\text{LIPS}_{n'} \equiv (2\pi)^4 \delta^4 \left(\sum_{i=1}^n k_i - \sum_{j=1}^{n'} k'_j \right) \prod_{l=1}^{n'} \frac{d^3 \mathbf{k}'_l}{(2\pi)^3 2k'_l{}^0} \quad (53)$$

In our case, we are considering the $n' = 2$ case. Then we get

$$\begin{aligned}
d\text{LIPS}_{2'} &= (2\pi)^4 \delta^4(p + k - p' - k') \frac{d^3 \mathbf{p}'}{(2\pi)^3 2p'^0} \frac{d^3 \mathbf{k}'}{(2\pi)^3 2k'^0} \\
&= \frac{1}{(2\pi)^2 4p'^0 k'^0} |\mathbf{k}'|^2 \left[\frac{d(p'^0 + k'^0 - p^0 - k^0)}{d|\mathbf{k}'|} \right]^{-1} d\Omega' \\
&= \frac{\omega'}{16\pi^2 E'} \left(\frac{d}{d\omega'} \left[(m^2 + \mathbf{p}'^2)^{1/2} + \omega' \right] \right)^{-1} d\Omega' \\
&= \frac{\omega'}{16\pi^2 E'} \left(\frac{d}{d\omega'} \left[(m^2 + (\mathbf{k} - \mathbf{k}')^2)^{1/2} + \omega' \right] \right)^{-1} d\Omega' \\
&= \frac{\omega'}{16\pi^2 E'} \left(\frac{d}{d\omega'} \left[(m^2 + \omega^2 + \omega'^2 - 2\omega\omega' \cos \theta')^{1/2} + \omega' \right] \right)^{-1} d\Omega' \\
&= \frac{\omega'}{16\pi^2 E'} \left(\frac{2\omega' - 2\omega \cos \theta + 2E'}{2E'} \right)^{-1} d\Omega' \\
&= \frac{\omega'}{16\pi^2 E'} \left(\frac{E'}{\omega' - \omega \cos \theta + E'} \right) d\Omega' \\
&= \frac{\omega'}{16\pi^2} \left(\frac{1}{\omega' - \omega \cos \theta + (m + \omega - \omega')} \right) d\Omega' \\
&= \frac{\omega'}{16\pi^2} \left(\frac{1}{m + \omega (1 - \cos \theta)} \right) d\Omega' \\
&= \frac{\omega'}{16\pi^2} \frac{\omega'}{m\omega} d\Omega'
\end{aligned} \tag{54}$$

The we get

$$\begin{aligned}
d\sigma &= \frac{1}{4m\omega} |\overline{\mathfrak{M}}|^2 d\text{LIPS}_{2'} \\
&= \frac{1}{4m\omega} |\overline{\mathfrak{M}}|^2 \frac{\omega'}{16\pi^2} \frac{\omega'}{m\omega} d\Omega' \\
&= \frac{\omega'^2}{4(4\pi)^2 m^2 \omega^2} |\overline{\mathfrak{M}}|^2 d\Omega' \\
&= \frac{\omega'^2}{(4\pi)^2 \omega^2} \frac{e^4}{2m^2} \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) d\Omega' \\
&= \frac{\alpha^2}{2m^2} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) d\Omega',
\end{aligned} \tag{55}$$

where $\alpha \equiv e^2/4\pi$ is the fine structure constant. Thus, you get the following Klein-Nishina formula

$$\frac{d\sigma}{d\Omega'}(\omega, \Omega') = \frac{\alpha^2}{2m^2} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right), \tag{56}$$

where ω' is uniquely determined by ω, Ω' as shown in Eq. (49).

Total Cross-Section

The total Klein-Nishina cross section is given by

$$\begin{aligned}
\sigma_{\text{KN}}(\omega) &= \int d\Omega' \frac{d\sigma}{d\Omega'} \\
&= \int_0^{2\pi} d\varphi \int_{-1}^1 d(\cos\theta) \frac{\alpha^2}{2m^2} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right) \\
&= \frac{\pi\alpha^2}{m^2} \int_{-1}^1 d(\cos\theta) \left(\frac{m}{m+\omega(1-\cos\theta)}\right)^2 \left\{ \left[\frac{m+\omega(1-\cos\theta)}{m}\right] + \left[\frac{m}{m+\omega(1-\cos\theta)}\right] - \sin^2\theta \right\} \\
&= \frac{\pi\alpha^2}{m^2} \int_{-1}^1 dt \left\{ \frac{1}{1+x(1-t)} + \left[\frac{1}{1+x(1-t)}\right]^3 - \left[\frac{1}{1+x(1-t)}\right]^2 + \left[\frac{t}{1+x(1-t)}\right]^2 \right\} \quad \left(x = \frac{\omega}{m}, t = \cos\theta\right) \\
&= \frac{\pi\alpha^2}{m^2} \int_{-1}^1 dt \frac{1+t^2 + (1+t^2)(1-t)x + (1-t)^2x}{[1+x(1-t)]^3} \\
&= \frac{\pi\alpha^2}{m^2} \left\{ \frac{2x[2+x(1+x)(8+x)]}{x^3(1+2x)^2} - \frac{[-x^2+2x+2]}{x^3} \log(1+2x) \right\} \\
&= \frac{2\pi\alpha^2}{m^2} \left\{ \frac{1+x}{x^3} \left[\frac{2x(1+x)}{1+2x} - \log(1+2x) \right] + \frac{1}{2x} \log(1+2x) - \frac{1+3x}{(1+2x)^2} \right\} \\
&= \frac{3}{4} \sigma_T \left\{ \frac{1+x}{x^3} \left[\frac{2x(1+x)}{1+2x} - \log(1+2x) \right] + \frac{1}{2x} \log(1+2x) - \frac{1+3x}{(1+2x)^2} \right\}, \tag{57}
\end{aligned}$$

where $\sigma_T = \frac{8\pi\alpha^2}{3m^2}$ is Thomson scattering cross section.

Loop Corrections

At the 1-loop level, the contributing Feynman diagrams include vertex corrections, vacuum polarization, and electron self-energy diagrams. The subgraphs included in these diagrams contribute to the renormalization of the electron mass and charge, modifying the total amplitude.